In Eq. (16), ℓ is the molecular mean free path, c the characteristic molecular velocity, and k the proportionality factor in the equation $\rho\ell = (2k)^{1/2}$. Therefore, given that $n = 10^0 \rightarrow 10^1$, the equilibration time, $\epsilon = n\ell/c$ enters the critical range [defined by Eq. (15)] in the standard atmosphere at an elevation of 3×10^4m, or in corresponding experimental conditions. Subsequent reductions in density further increase the observable nonequilibrium effects.

In a different transport modeling context, application of the relaxation equation to turbulent flows also has intuitive appeal, inasmuch as the elements of the Reynolds stress tensor are governed by convective equations. In this case, the formulation utilized here closely resembles currently evolving transport theories, such as the one developed by Bradshaw,[8] which also result in a hyperbolic system of equations.

This particular turbulence modeling approach does not require support from the physics of higher order kinetic theory. However, their resultant similar mathematical nature brings some unity to the consequences of adopting a particular class of transport model. Quite often, the modeling of physical processes leads naturally to a hyperbolic system of governing equations.[9] It is then desirable to expose the physical source of the limiting (often parabolic) mathematical nature of commonly used approximate transport models. Subsequently, it is not nearly so surprising when rigorous hyperbolic models are resurrected for particular applications.

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Criteria for Asymptotic Supersonic Two-Dimensional Turbulent Wakes

Michael J. Walsh*

NASA Langley Research Center, Hampton, Va.

Nomenclature

C_D = body drag coefficient,

$$\frac{2}{H}\int_{y=-\infty}^{\infty}\frac{\rho u}{\rho_\infty u_\infty}(u_\infty - u)\,\mathrm{d}y$$

H = base height of wedge
L = transverse length scale
Re_H = Reynolds number based on base height of the wedge
T = static temperature
u = longitudinal velocity
$\tilde{u}$ = similarity variable for velocity, $(u_\infty - u)/(u_\infty - u_0)$
x = longitudinal distance downstream of the wedge
y = transverse coordinate
$y_{1/2}$ = wake half-width
η = similarity variable for y coordinate, y/L
θ = temperature defect, $(T_0 - T_\infty)/T_\infty$
ρ = gas density
ω = velocity defect, $(u_\infty - u_0)/u_\infty$

Subscripts

e = edge of viscous wake
∞ = freestream
0 = centerline
T = quantity determined from temperature profiles

Superscript

$*$ = physical variable transformed to Howarth-Doroditzen coordinates

SEVERAL sets of experimental data exist to validate turbulence models for two-dimensional supersonic turbulent wakes.[1-7] To use these data for turbulence modeling requires determination of the extent of the fully developed turbulent region. Birch and Eggers[8] note that for a turbulent flow to be fully developed, the mean as well as the turbulence components must be similar and an asymptotic growth rate attained. In the past, the location of the fully developed turbulent flow in two-dimensional supersonic wakes has not been that well defined. References 4, 6, and 7 used similarity variables of the mean flow, while Ref. 3 examined the wake growth rates. Demetriades[7] found that the velocity profiles were not similar until the flow was fully developed turbulent; however, Wagner[4] found that his profiles were similar a considerable distance upstream of the point where he would expect the mean turbulent wake flow to be fully developed.

The purpose of this Note is to examine the similarity variables that have been used in the past to determine fully developed mean flow for supersonic two-dimensional wakes and to demonstrate that wake growth rates are better indicators that the mean flow is fully developed.

Demetriades[6] used the following similarity variables:

$$\tilde{u} = (u_\infty - u)/(u_\infty - u_0) \tag{1}$$

and

$$\eta = y^*/L \tag{2}$$

where the transverse length scale is related to the body drag coefficient C_D as follows

$$L = C_D H/4\omega \tag{3}$$

The variable H is the base height of the wedge, ω is the velocity defect, and u_∞ and u_0 are the freestream and centerline velocities, respectively. The variable y^* is the physical y distance transformed to Howarth-Dorodnitzen coordinates using the following equation:

$$y^* = \int_0^y \frac{\rho}{\rho_\infty}\,\mathrm{d}y \tag{4}$$

Because he used a slender wake generator, Demetriades[6] was able to neglect the recompression and expansion waves present in a supersonic wake and assume that the properties at

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*Aerospace Engineer, Fluid Mechanics Branch, High-Speed Aerodynamics Division. Member AIAA.

the edge of the viscous wake were equal to those in the freestream. The analysis that follows also assumes that the edge values are equal to freestream values. This procedure should be good for small angle wake generators, but it needs to be examined for large angle wake generators if it is found that the similarity variables change as the wake generator angle changes.

Using an average value of $C_D H$ to determine the transverse wake scale in Eq. (3), Demetriades[6,7] found deviations from similarity near the wedge base; however, the data seem to indicate that these deviations were due to x variations in $C_D H$ near the wedge, rather than the profiles not being similar. When the appropriate x values of $C_D H$ were used to determine the transverse wake scale, the deviations from similarity decreased.

Demetriades[6,7] also found that the transverse length scale, as defined in Eq. (3), was approximately equal to the wake half-width, $y^*_{½}$, defined below

$$y^*_{½} = y^* \qquad \text{where } \tilde{u} = 0.5 \tag{5}$$

The data of Demetriades[6] and Wagner[4] are plotted in Fig. 1 in terms of similarity variables where the chosen length scales are the wake half-widths, $y_{½}$ and $y^*_{½}$. Note that the edge value of velocity is used for the freestream value. The data indicate that the wake half-width is a good length scale to collapse the velocity profiles in terms of similarity parameters as long as consistent values are used (i.e., $y/y_{½}$ and $y^*/y^*_{½}$). The data also show that u_e can be used instead of u_∞ for the large angle wake generator used in Ref. 4. It will be shown later that the data of Demetriades[6] presented in Fig. 1 extend from the laminar to fully developed turbulent region. Therefore, the experimental data confirm that the velocity profiles when nondimensionalized in the usual manner as just shown appear to be similar whether or not the flow is laminar, transitional, or turbulent.

An analysis similar to that for the velocity profiles can be applied to the static temperature profiles. If the local temperature T, and physical coordinate y, are nondimensionalized as follows,

$$\tilde{T} = (T - T_\infty)/(T - T_\infty) \tag{6}$$

and

$$\eta_T = y/L_T \tag{7}$$

the temperature profiles are similar.

Figure 2 shows available temperature profiles obtained in Refs. 3 and 6 in similarity variables using the wake half-width based on the temperature profiles as the scaling parameter. Note some of the profiles obtained in Refs. 3 and 6 were taken in the transition region of the wake. The data indicate that the similarity variables $y/y_{½,T}$ and $y^*/y^*_{½,T}$ collapse all the data fairly well. There is no evidence that the profiles are similar only in the fully developed turbulent region. Also shown on Fig. 2 is the similarity variable used by Demetriades[6,7] (in Refs. 6 and 7, $L \approx y_{½}$) and Wagner[4] for temperature profiles. As expected, the data scatter is increased when $y/y_{½}$ is used. The only time that $y_{½}$ will collapse the temperature profile data is if the temperature and velocity defect were inversely proportional to the same power of x.

As previously shown, velocity and temperature profiles in similarity variables do not adequately distinguish transition from fully developed turbulent flow. Next, another method of determining the extent of fully developed turbulent flow is discussed.

Halleen[9] and Kline[10] have shown that for incompressible flow the growth rate of the two-dimensional turbulent wake is proportional to the square root of distance. Demetriades[6] at Mach 3 and Walsh[3] at Mach 6 have found the same growth rate for compressible two-dimensional wakes. It is of interest to examine the growth rate as the wake goes from laminar to turbulent flow to see if changes in the growth rate can be used to locate the onset of fully developed turbulent wake flow.

Figure 3 shows the growth of the supersonic two-dimensional wakes that have been previously reported. The shaded symbols are the first data points past the transition

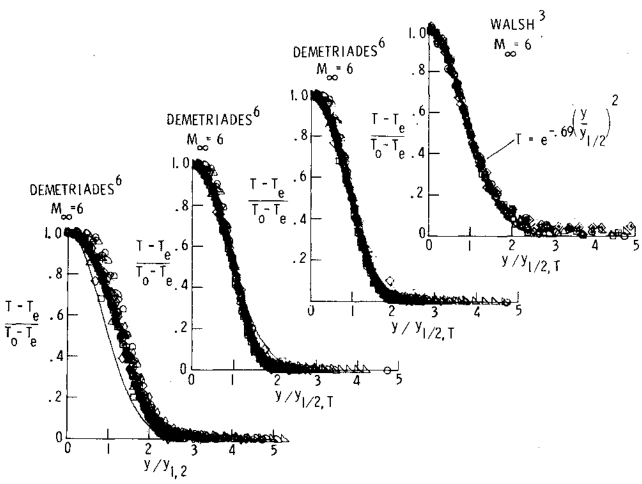

Fig. 2 Two-dimensional wake temperature profiles in terms of similarity variables.

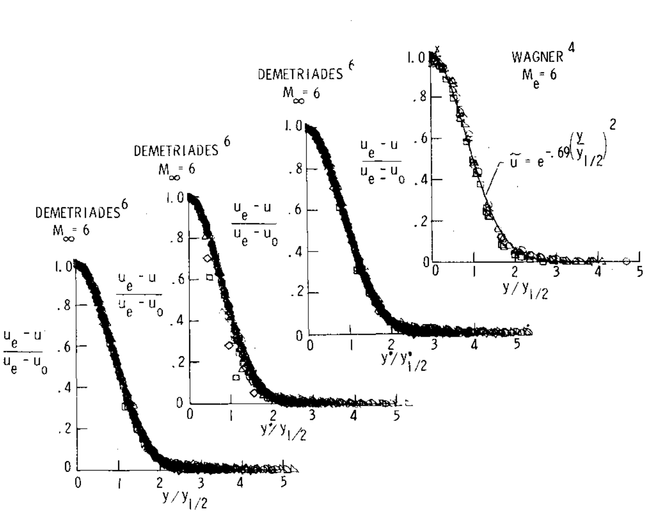

Fig. 1 Two-dimensional wake velocity profiles in terms of similarity variables.

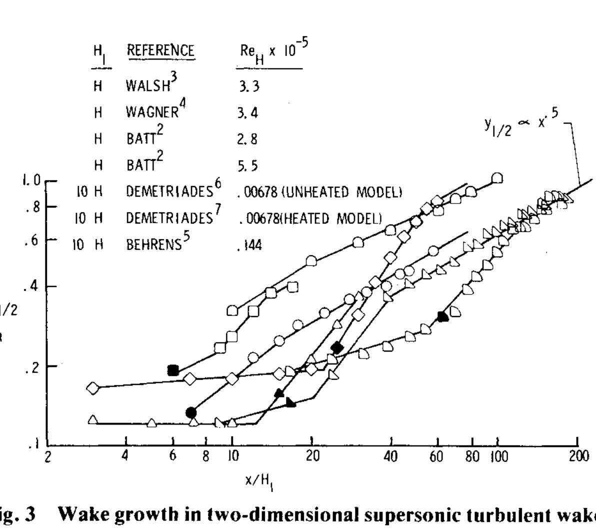

Fig. 3 Wake growth in two-dimensional supersonic turbulent wakes.

location determined by using the velocity defect (see Ref. 6). The data indicate that the growth of the wake is small in the laminar region, increases at transition, and that it finally levels off to the growth rate found previously for incompressible fully developed turbulent flows. This is particularly clear in the data of Walsh,[3] Behrens et al.,[5] and Demetriades.[6,7] In summary, Fig. 3 shows that the wake growth rate variation is a better indicator of the onset of fully developed turbulent flow than similarity variables used previously.

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Exact Similar Solution for an Axisymmetric Laminar Boundary Layer on a Circular Cone

B. R. Clayton* and B. S. Massey†

University College London, England

ALTHOUGH approximate similar solutions are available for a laminar boundary layer on a circular cone,[1] the mainstream velocity distribution is then linked to the vertex angle of the cone. However, an exact solution of the Navier-Stokes equations with an explicit expression for c_f is obtained when, for any vertex angle, the mainstream velocity is inversely proportional to the distance from the vertex.

Using spherical polar coordinates centered at the vertex of the cone, a velocity component u is defined as positive in the direction of increasing radial distance r; v is considered positive in the direction of increasing θ (the semi-angle of the cone being θ_0); there is no circumferential velocity. An irrotational main stream and constant density ρ and kinematic viscosity ν are assumed.

The Navier-Stokes equation[2] for the θ direction is multiplied by r, then differentiated with respect to r and subtracted from the θ derivative of the equation for the r direction. A stream function ψ is introduced, defined by

$$u = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}; \quad v = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

If the resulting equation is to yield similar solutions and the transverse curvature of the surface is accounted for, the boundary-layer thickness δ at a particular position must be proportional to the radius of the surface there, that is to $r\sin\theta_0$ (unless $\theta_0 = \pi/2$, in which case the surface has no curvature). Hence δ/r = constant and so δ may be expressed simply in terms of $\theta - \theta_0$; that is, θ may be regarded as the similarity variable, and we set $\psi = \nu\ell f(\theta)$, where ℓ is a function of r only.

The equation of motion then becomes

$$\begin{aligned}
&\frac{2c}{r^2 s^3}\ell_r \ell_{rr} f^2 - ff'\left\{\frac{1+2c^2}{r^4 s^4}\ell\ell_r + \frac{2}{r^3 s^2}\ell\ell_{rr}\right. \\
&\quad \left. - \frac{1}{r^2 s^2}\ell\ell_{rrr} + \frac{1}{r^2 s^2}\ell_r \ell_{rr}\right\} + \frac{3c}{r^4 s^3}\ell\ell_r ff'' - \frac{1}{r^4 s^2}\ell\ell_r ff''' \\
&\quad + (f')^2\left\{\frac{4c}{r^5 s^3}\ell^2 - \frac{c}{r^4 s^3}\ell\ell_r\right\} + f'f''\left\{\frac{1}{r^4 s^2}\ell\ell_r - \frac{4}{r^5 s^2}\ell^2\right\} \\
&\quad = -\frac{1}{s}\ell_{rrrr} f + f'\left\{\frac{c(6s^2+3)\ell}{r^4 s^4} - \frac{4c\ell_r}{r^3 s^2} + \frac{2c\ell_{rr}}{r^2 s^2}\right\} \\
&\quad + f''\left\{\left(\frac{5c^2-8}{r^4 s^3}\right)\ell + \frac{4\ell_r}{r^3 s} - \frac{2\ell_{rr}}{r^2 s}\right\} + \frac{2c\ell}{r^4 s^2}f''' - \frac{\ell}{r^4 s}f'''' \qquad (1)
\end{aligned}$$

where $c = \cos\theta$, $s = \sin\theta$, the primes denote differentiation with respect to θ, and the suffixes denote differentiation with respect to r.

To make the coefficients of f''' and f'''' independent of r, the equation must be multiplied by r^4/ℓ. Then the coefficient of ff''', for example, becomes $-\ell_r/s^2$. Consequently, similar solutions require ℓ_r to be independent of r. The coefficient of $(f')^2$ becomes

$$\frac{4c\ell}{s^3 r} - \frac{c}{s^3}\ell_r$$

and so ℓ/r must also be independent of r. Thus $\ell \propto r$, and, as the coefficient of proportionality may be absorbed into the definition of f, we set $\ell = r$. Equation (1) therefore becomes

$$\begin{aligned}
&3\frac{c}{s^2}(f')^2 - \frac{3}{s}f'f'' - \left(\frac{1+2c^2}{s^3}\right)ff' + \frac{3c}{s^2}ff'' - \frac{1}{s}ff''' \\
&\quad = \frac{c}{s^3}(2s^2+3)f' + \left(\frac{c^2-4}{s^2}\right)f'' + \frac{2c}{s}f''' - f''''
\end{aligned}$$

Changing the variable from $f(\theta)$ to $F(c)$ yields

$$3F'F'' + FF''' = 4cF''' - (1-c^2)F'''' \qquad (2)$$

where the primes now denote differentiation with respect to c.

Equation (2) is essentially that obtained by Morgan[3] for flow between two coaxial cones with a common vertex, but he did not consider the boundary-layer problem.

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*Lecturer, Dept. of Mechanical Engineering.

†Reader, Dept. of Mechanical Engineering.